Grade 7/8 Math Circles
November 20/21/22/23, 2023
Parabolas - Solutions

Exercise Solutions

Exercise 1

Answer the following questions with this activity https://www.geogebra.org/m/fw9u6zRG.

(a) If you increase the distance between the directrix and the focus, what happens to the parabola's shape?

(b) What about if you decrease the distance between the directrix and the focus?

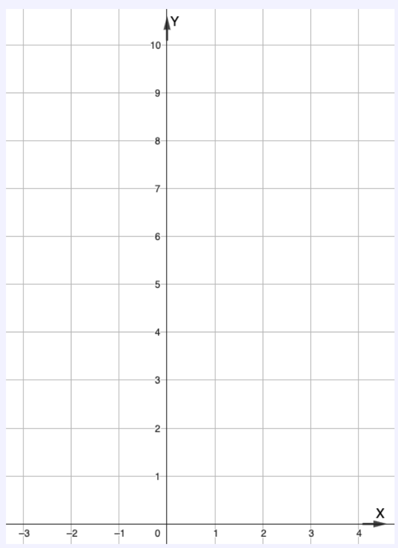
Exercise 1 Solution

As you increase the distance between the directrix and the focus, the parabola becomes wider. The parabola gets narrower when the directrix and the focus become closer.

Exercise 2

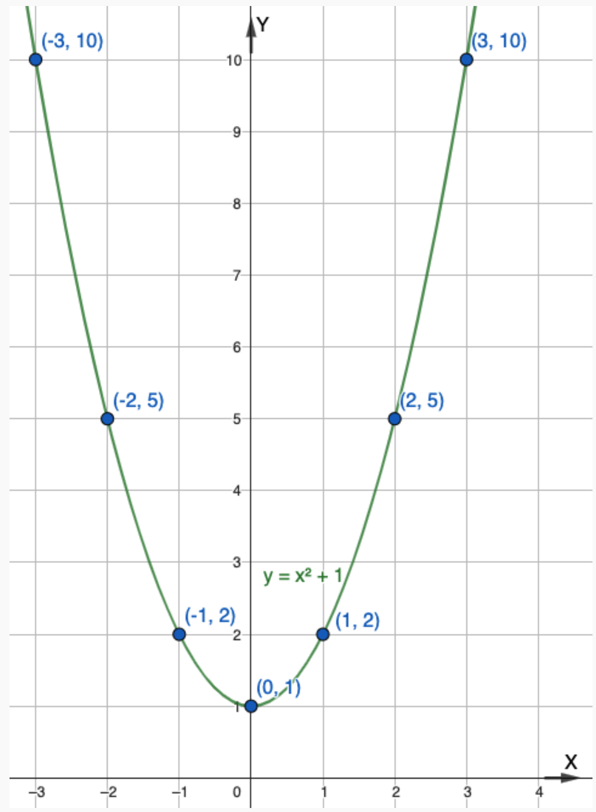
Graph the function $y = x^2 + 1$ using the table and graph below.

x	$y = x^2 + 1$	Coordinate (x, y)
-3		
-2		
-1		
0		
1		
2		
3		



Exercise 2 Solution

x	$y = x^2 + 1$	Coordinate (x, y)
-3	10	$(-3, 10)$
-2	5	$(-2, 5)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	5	$(2, 5)$
3	10	$(3, 10)$



Exercise 3

Use the slider from above (linked again here; https://www.geogebra.org/m/wvpjqrry) to answer the following questions.

(a) When does the parabola open up. When does it open down?

(b) How does sliding a affect the parabola? How is this related to changing the distance between the focus and directrix?

(c) How does sliding c affect the parabola?

Exercise 3 Solution

(a) The parabola opens up when $a > 0$. The parabola opens down when $a < 0$.

(b) As a gets further away from zero (both in the positive and negative directions), the parabola becomes narrower. As a gets closer to zero, the parabola becomes wider. This is similar to changing the distance between the focus and directrix. Increasing the distance between the focus and directrix makes the parabola wider, which also happens when a gets close to zero.

(c) Increasing c shifts the entire parabola up and decreasing c shifts the entire parabola down.

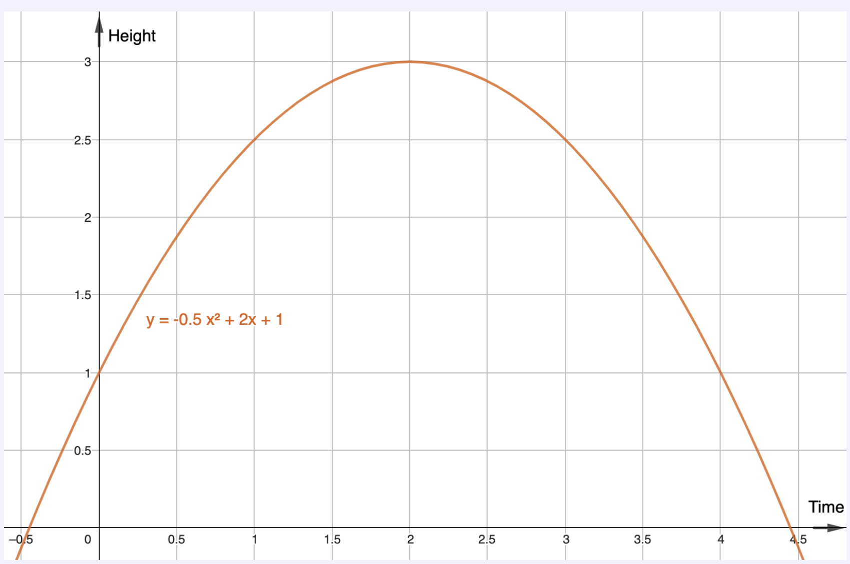
Exercise 4

Suppose the path of a football can be modelled by the function $h = -0.5t^2 + 2t + 1$. Using the graph of this function, answer the questions below. Note that the horizontal axis is the time in seconds and the vertical axis is the height of the ball in metres.

(a) What is the height of the ball at time $t = 0$? Think of a reason the ball starts at this height.

(b) Determine at what time the ball reaches its maximum height, and determine the ball's maximum height.

(c) Determine when the ball hits the ground.



Exercise 4 Solution

1. The height of the ball at time $t = 0$ is 1 m. This is likely because the football players is throwing the ball from their shoulder.
2. The maximum height of the ball is 3 m and this occurs 2 seconds after the player throws the ball.
3. The ball hits the ground just before 4.5 seconds.

Problem Set Solutions

1. For the following functions, fill in the table and plot the points from the table on a Cartesian-plane. Determine if the graph forms a parabola.

(a)

x	$y = -2x^2 + 3$	(x, y)
-3		
-2		
-1		
0		
1		
2		
3		

(b)

x	$y = x^4 - 1$	(x, y)
-3		
-2		
-1		
0		
1		
2		
3		

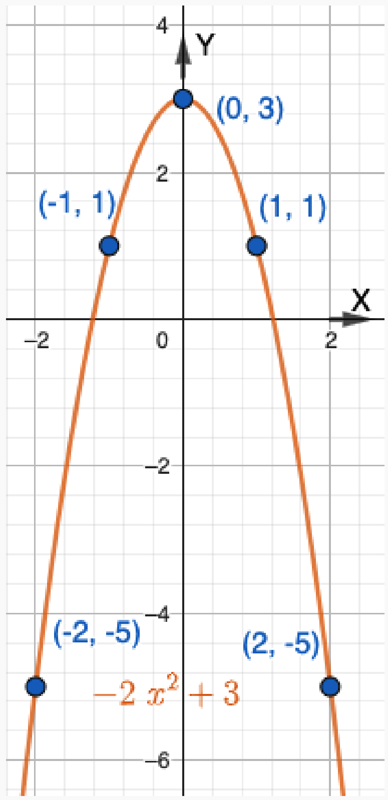
Solution:

(a)

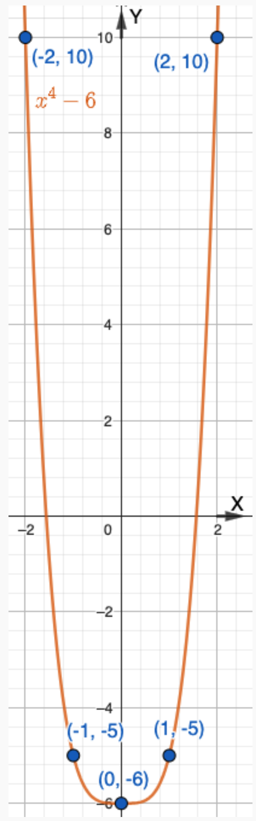
x	$y = -2x^2 + 3$	(x, y)
-2	-5	$(-2, -5)$
-1	1	$(-1, 1)$
0	3	$(0, 3)$
1	1	$(1, 1)$
2	-5	$(2, -5)$

(b)

x	$y = x^4 - 6$	(x, y)
-2	10	$(-2, 10)$
-1	-5	$(-1, -5)$
0	-6	$(0, -6)$
1	-5	$(1, -5)$
2	10	$(2, 10)$



This is a parabola because it can be written in the form $ax^2 + bx + c$.



This is not a parabola because parabolas don't contain terms with x^4.

2. Answer the following questions about these these four functions **without** graphing.

(1) $y = -x^2 + 5$ (2) $y = 2x^2 - 1$ (3) $y = -4x^2$ (4) $y = 0.5x^2 + x$

(a) Determine whether the parabolas of the above functions open up or down.

(b) Order the functions from narrowest to widest.

Solution:

(a) (1) opens down. (2) opens up. (3) opens down. (4) open up.

This is determined by the sign of the x^2 term. If it's positive, it opens up. If it's negative, it opens down.

(b) The order from narrowest to widest is (3), (2), (1), (4).

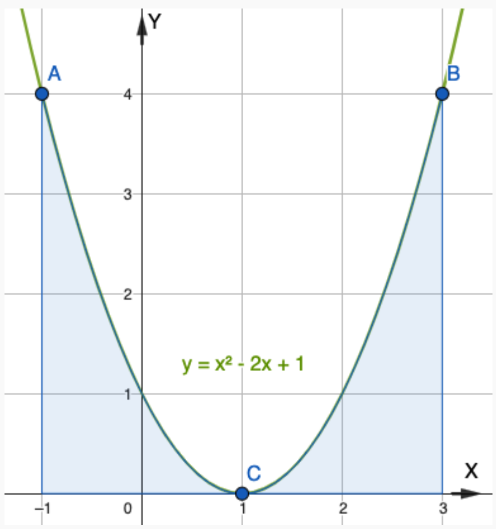
This is determined by the coefficient on the x^2 term. As the coefficient gets further from zero, the parabola gets narrower. As it gets closer to zero, the parabola gets wider.

3. Calculate the area under the parabolas from the lower bound to the upper bound on the x-axis. Hint: Start by graphing the function.

   (a) $y = x^2 - 2x + 1$ from $x = -1$ to $x = 3$

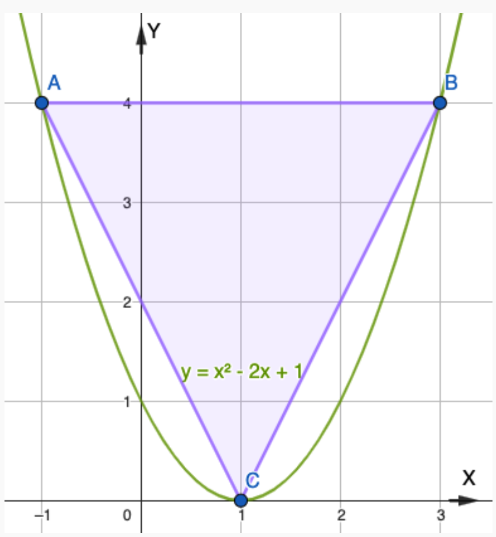
   (b) $y = 0.5x^2 + 2x + 2.5$ from $x = -5$ to $x = 1$

Solution:

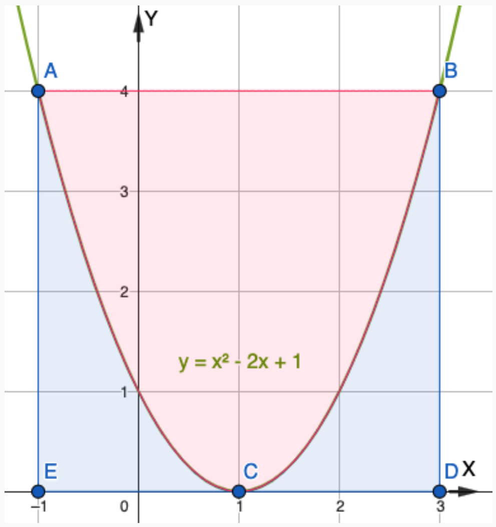
(a) 

The green line is the graph of the function. The blue region is the area we are calculating.

This solution follows the steps outlined in the lesson.



The purple triangle has a base of 4 and a height of 4 so $A_{ABC} = 8$.



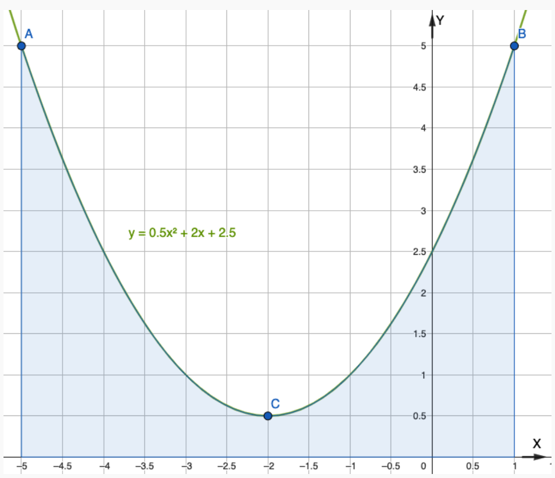
The red region is four-thirds of the triangle's area. So, $A_{red} = \frac{4}{3} \times 8 = \frac{32}{3}$.

$A_{ABDE} = 4 \times 4 = 16$.

Now subtract the red area from the area of square ABDE to get the blue area.

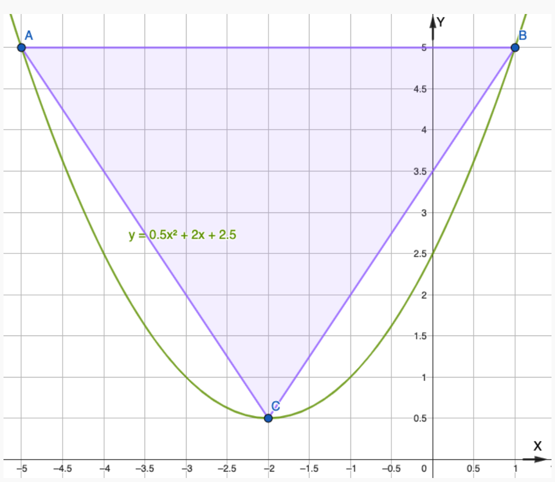
$A_{blue} = 16 - \frac{32}{3} = \frac{16}{3}$

(b)

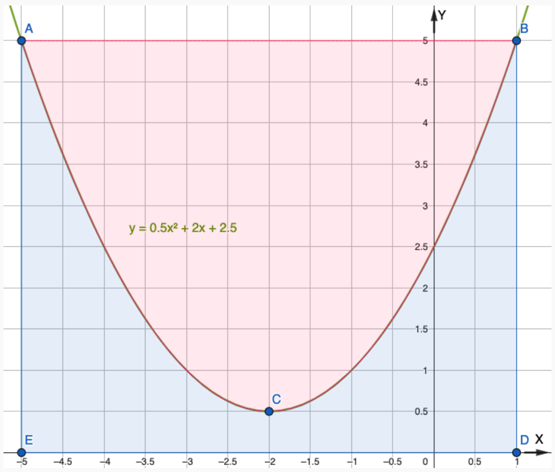


The green line is the graph of the function. The area of the blue region is the area we are calculating.

This solution follows the steps outlined in the lesson.



The purple triangle has a base of 6 and a height of 4.5 so $A_{ABC} = \frac{27}{2}$.



The red region is four-thirds of the triangle's area. So, $A_{red} = \frac{4}{3} \times \frac{27}{2} = 18$.

$A_{ABDE} = 6 \times 5 = 30$.

Now subtract the red area from the area of square ABDE to get the blue area.

$A_{blue} = 30 - 18$

$A_{blue} = 12$

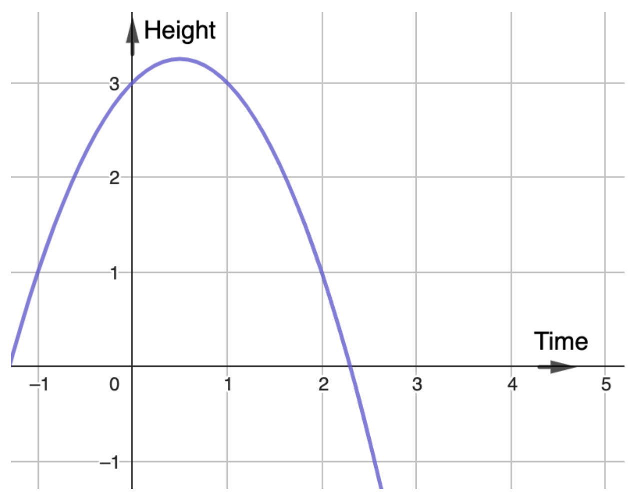
4. There's a pirate battle on the horizon! Captain Longbeard is ready to fire his cannon at the ship of the notorious Captain Redbeard! Longbeard knows that the cannonball must be in the air for exactly 4 seconds to make a direct hit. The cannon is always fired at time $t = 0$.

   (a) The arch that the cannonball follows can be described by the function $h = -t^2 + 4t + 3$, where h is the height above the sea in meters at time t in seconds.

   Here is the graph of the first cannonball's flight path.

   From what height was the first cannonball launched?

   Did the first shot stay in the air for the correct amount of time to hit Captain Redbeard's ship?

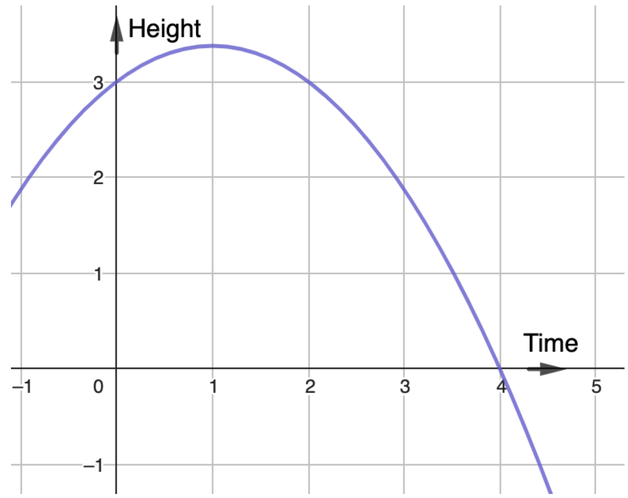


(b) For the second shot, reset the timer so the shot occurs at time $t = 0$. The arch of this cannonball follows the function $h = -\frac{3}{8}x^2 + \frac{3}{4}x + 3$.

Here is a graph of the second cannonball's light path.

Did the second shot stay in the air for the correct amount of time to hit Captain Redbeard's ship?

What is the maximum height of the second cannonball and when did it occur?



Solution:

(a) Look at the graph from part (a) to determine the following information.

At $t = 0$, $h = 3$ so the first cannonball was launched from 3 m above the sea.

The cannonball hits the water between 2 and 3 seconds, so it wasn't in the air for long enough to hit the ship.

(b) Look at the graph from part (b) to determine the following information.

At $t = 4$, $h = 0$ so the cannonball was in the air for 4 seconds! DIRECT HIT!

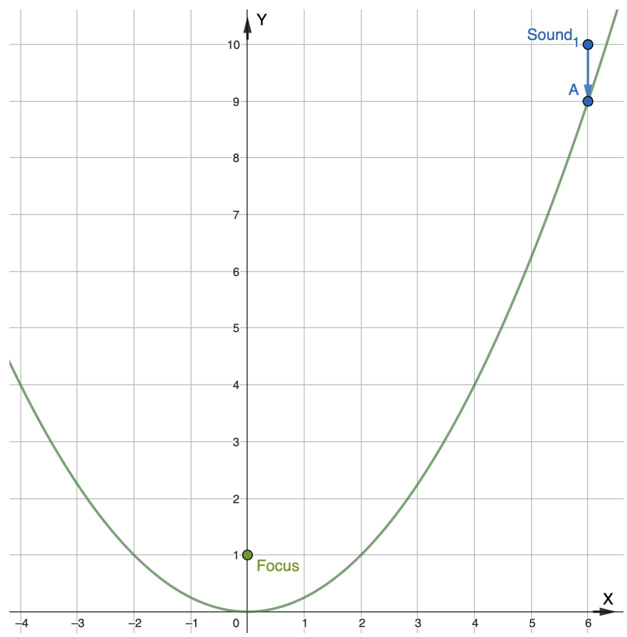
By looking at the graph, notice that the maximum height occurs at time $t = 1$. Then by plugging in $t = 1$ into the height formula, we get the height at $t = 1$ is $-\frac{3}{8}(1)^2 + \frac{3}{4}(1) + 3 = \frac{27}{8}$.

5. Sound amplifiers use a parabolic shape to gather sound waves by reflecting them to a microphone at the focus of the parabolic shield.

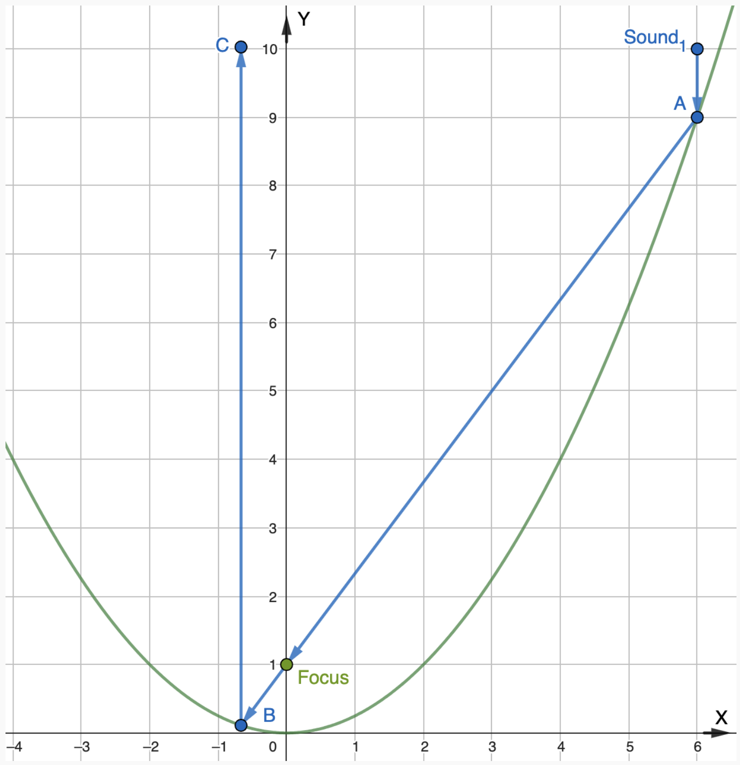
   But what if someone forgot to install the microphone? How would the sound waves behave after bouncing off the shield? Answer this by completing the paths of the sound wave on the right. Then make an observation about the path of the wave.

   HINT: No microphone is at the focus so the wave will continue straight passed the focus.

Image retrieved from Amazon



Solution:



The sound waves first reflect and pass through the focus (as we saw in the lesson). Then they reflect off another part of the parabola and leave parallel to the y-axis. This will happen for waves entering the parabola at *any* position.

6. CHALLENGE QUESTION 1

Recall that integrals like $\int_1^{2.5}(x^2+3x-4)\,dx$ can be used to calculate the area under functions. This one calculates the area under x^2+3x-4 from $x=1$ to $x=2.5$.

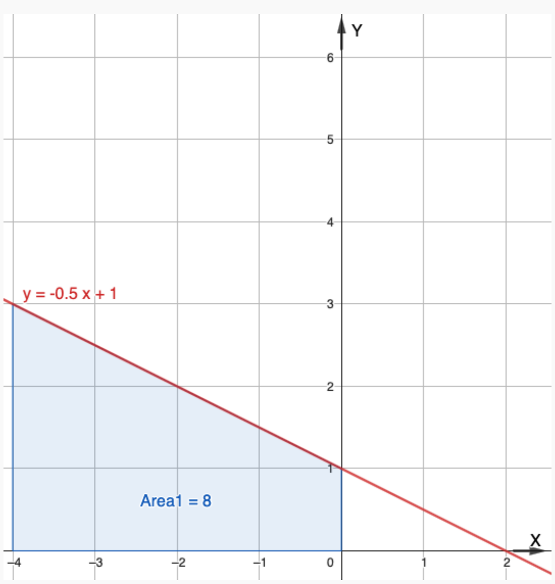
Graph the function and shade in the area that the following integrals calculate.
NOTE: You don't need to calculate the area, just shade the region. (But you can calculate the area using basic geometry for part (a) and Archimedes' method for part (b)).

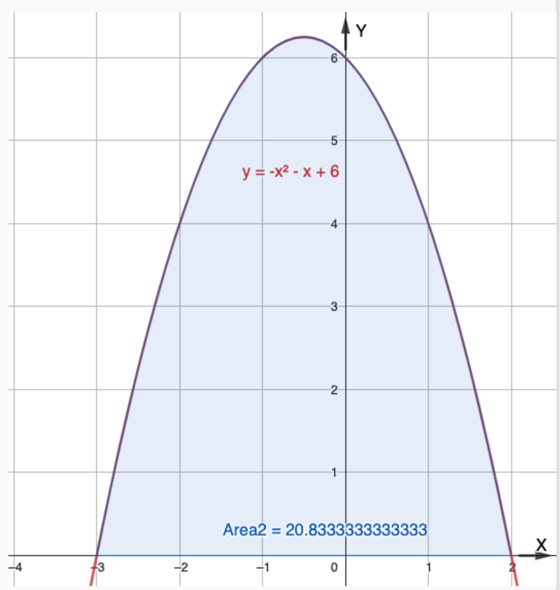
(a) $\int_{-4}^{0}(-0.5x+1)\,dx$
(b) $\int_{-3}^{2}(-x^2-x+6)\,dx$

Solution:

(a) This integral calculates the area under $y=-0.5x+1$ from $x=-4$ to $x=0$. The shaded region is on the right. The area is equal to 8.

(b) This integral calculates the area under $y=-x^2-x+6$ from $x=-3$ to $x=2$. The shaded region is on the right. The area is equal to $\dfrac{125}{6}$.



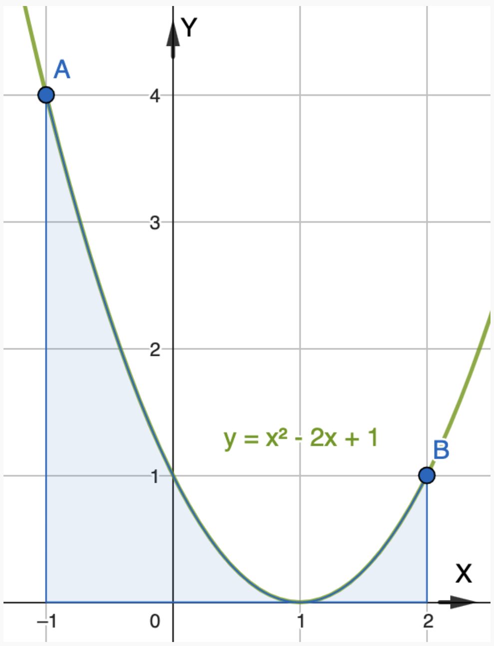


7. CHALLENGE PROBLEM 2

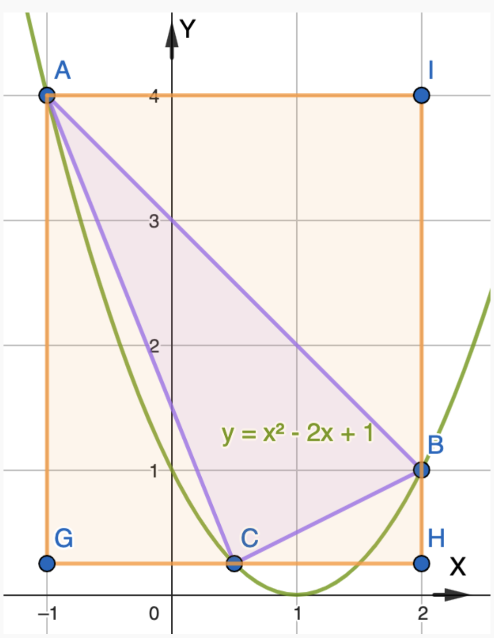
Find the area under the parabola $y = x^2 - 2x + 1$ from the $x = -1$ to $x = 2$.

HINT: If you did question 3(a), then you have already graphed this function!

Solution:



The green line is the graph of the function. The blue region is the area we are calculating.
This solution follows the steps outlined in the extension.
Points A and B are found by plugging $x = -1$ and $x = 2$ into the function.
Point C is halfway between -1 and 2, so point C is $(0.5, 0.25)$.



To find the area of triangle ABC, subtract the area of each orange triangle from the area of rectangle AGHI.

$A_{AGHI} = \frac{15}{4} \times 3 = \frac{45}{4}$

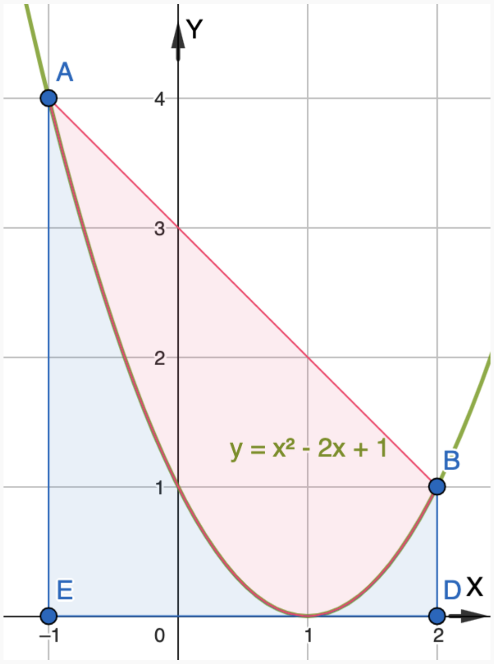
$A_{ACG} = \dfrac{\frac{15}{4} \times \frac{3}{2}}{2} = \frac{45}{16}$

$A_{BCH} = \dfrac{\frac{3}{4} \times \frac{3}{2}}{2} = \frac{9}{16}$

$A_{ABF} = \dfrac{3 \times 3}{2} = \frac{9}{2}$

$A_{ABC} = \frac{45}{4} - \frac{45}{16} - \frac{9}{16} - \frac{9}{2} = \dfrac{27}{8}$

The purple triangle has an area of $\dfrac{27}{8}$.



The red region is four-thirds of the triangle's area. So, $A_{red} = \frac{4}{3} \times \frac{27}{8} = \dfrac{9}{2}$.

$A_{blue} = (\frac{4+1}{2} \times 3) - \frac{9}{2}$.

$A_{blue} = \frac{15}{2} - \frac{9}{2}$

$A_{blue} = 3$